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NS5-branes, T-duality and worldsheet instantons

David Tong

Center for Theoretical Physics, Massachusetts Institute of Technology
Cambridge, MA 02139, U.S.A.
E-mail: `dtong@lns.mit.edu`

Abstract: The equivalence of NS5-branes and ALF spaces under T-duality is well known. However, a naive application of T-duality transforms the ALF space into a smeared NS5-brane, de-localized on the dual, transverse, circle. In this paper we re-examine this duality, starting from a two-dimensional $\mathcal{N} = (4, 4)$ gauged linear sigma model describing Taub-NUT space. After dualizing the $\mathbf{S}^1$ fiber, we find that the smeared NS5-brane target space metric receives corrections from multi-worldsheet instantons. These instantons are identified as Nielsen-Olesen vortices. We show that their effect is to break the isometry of the target space, localizing the NS5-brane at a point. The contribution from the k-instanton sector is shown to be proportional to the weighted integral of the Euler form over the k-vortex moduli space. The duality also predicts the, previously unknown, asymptotic exponential decay coefficient of the BPS vortex solution.

Keywords: Field Theories in Lower Dimensions, Solitons Monopoles and Instantons, String Duality.

Contents

1. Introduction and summary

Under T-duality of type II string theory, NS5-branes are mapped into Ricci flat, background geometries. For N parallel NS5-branes, this background is the hyperKähler metric on an asymptotically locally flat (ALF) space with an A_{N-1} singularity [1]. This relationship plays a prominent role in the duality plexus yet, from the perspective of the string worldsheet, has been proven only for *smeared* NS5-branes, for which the transverse circle is an isometry and the usual Buscher rules for T-duality may be applied [2, 3]. This is an unsatisfactory state of affairs. The purpose of this paper is to rectify this situation and demonstrate T-duality between the ALF space and the *localized* NS5-brane. As we shall see, the missing ingredient is the contribution from worldsheet instantons.

Our tool in exploring this duality is the $\mathcal{N} = (4,4)$ supersymmetric gauged linear sigma model for the ALF space. Since their inception, linear sigma models have proven useful in determining the effects of worldsheet instantons [4, 5]. More pertinently, Hori and Vafa have recently used this technique to calculate instanton corrections to T-duality transformations in theories with $\mathcal{N} = (2,2)$ supersymmetry [6]. Subsequent applications include [7]. In each of these cases, a superpotential is generated by a one instanton effect, breaking a global symmetry of the theory. While the calculations presented below are similar to those of [6, 7], they differ in two important respects. Firstly, the existence of $\mathcal{N} = (4,4)$ supersymmetry prohibits the generation of a superpotential, and the instantons now correct the target space metric where their effect is to break an isometry. Secondly, the corrections do not stop at the one instanton level, and the metric receives contributions from all topological sectors.

To make this discussion more explicit, let us start with type II string theory "compactified" on an ALF space. For simplicity we consider Taub-NUT space. It is worth noting that this is T-dual to a single NS5-brane, and subtleties abound concerning this object [8]. However, these difficulties play no role in the following discussion. The metric of Taub-NUT space is given by,

$$ds^2_{ALF} = \left(\frac{1}{g^2} + \frac{1}{2r}\right) d\mathbf{r}\cdot d\mathbf{r} + \frac{1}{4}\left(\frac{1}{g^2} + \frac{1}{2r}\right)^{-1} (d\psi + \boldsymbol{\omega}\cdot d\mathbf{r})^2 \tag{1.1}$$

where $\mathbf{r} \in \mathbf{R}^3$ and $\psi \in [0, 4\pi)$ and $\nabla \times \boldsymbol{\omega} = \nabla(1/r)$. The harmonic function $H(r)$ is given by,

$$H(r) = \frac{1}{g^2} + \frac{1}{2r} \tag{1.2}$$

As $r \to 0$, the metric becomes flat $\mathbf{R}^4$ while, in the asymptotic regime $r \to \infty$, the metric is locally $\mathbf{R}^3 \times \mathbf{S}^1$, with the $\mathbf{S}^1$ parameterized by ψ. The asymptotic radius of this circle is g.

This metric enjoys a U(1) isometry which acts by shifting the value of ψ, ensuring that momentum around the circle is a conserved quantum number. In contrast, string winding number around the circle is not conserved since the string may slip off, either by moving to $r = 0$ where the circle degenerates, or alternatively by wrapping once around the asymptotic boundary of Taub-NUT [9].

The existence of the U(1) isometry allows for the application of the usual T-duality transformation, exchanging momentum and winding on the string worldsheet: $\partial_\mu\psi = g^2\epsilon_{\mu\nu}\partial^\nu\theta$. As we will review in section 2, after such an operation the resulting metric has the form,

$$ds^2_{NS5} = H(r)\left(d\mathbf{r}\cdot d\mathbf{r} + d\theta^2\right) \tag{1.3}$$

where $\theta \in [0, 2\pi)$ parameterizes the dual $\mathbf{S}^1$. The conformal factor H is related to the dilaton and is given by (1.2), implying that the dual circle has asymptotic radius $1/g$, as expected under T-duality. Moreover, as we shall see explicitly in section 2, the non-trivial fibration of $\mathbf{S}^1$ over $\mathbf{R}^3$ in Taub-NUT space results in a torsion term, $T_{ijk} = \epsilon^l_{ijk}\partial_l H^{-1}$, carrying the charge of a single NS5-brane. However, as may be seen from (1.2) and (1.3), the metric has no dependence on the dual circle θ. In other words, the NS5-brane has been smeared in this direction.

Thus, after the naive application of T-duality, we find a string background in which both momentum and winding are conserved. Let us contrast this situation with the metric for an NS5-brane that is fully localized on $\mathbf{R}^3 \times \mathbf{S}^1$. This can be easily determined from supergravity by considering an infinite, periodic, array of colinear NS5-branes. Upon Poisson resummation, the metric takes the form (1.3), but with the smeared function (1.2) replaced by [9]

$$H_{\text{sugra}}(r,\theta) = \frac{1}{g^2} + \frac{1}{2r}\left(\frac{\sinh r}{\cosh r - \cos\theta}\right) \tag{1.4}$$

Notice in particular the appearance of θ in the function $H_{\rm sugra}$, ensuring that the circle transverse to the NS5-brane is no longer an isometry,[1] and momentum in this direction is no longer conserved.

Our challenge is to reproduce this localization of the NS5-brane from the worldsheet perspective of T-duality. A clue as to the mechanism responsible for this localization can be found from the Taylor expansion [9],

$$H_{\rm sugra}(r,\theta)=\frac{1}{g^2}+\frac{1}{2r}\left(1+\sum_{k=1}^{\infty}\sum_{\pm}e^{-kr\pm ik\theta}\right) \tag{1.5}$$

which points the finger of responsibility firmly at worldsheet instantons. Thus, in section 3 we turn to the task of uncovering the instanton structure of the gauge theory. As we shall see, although the theory does contain the topology necessary to admit instantons, there are in fact no finite action solutions to the equations of motion. The situation is reminiscent of constrained instantons in four-dimensional Yang-Mills-Higgs theories [10], and we proceed accordingly. The lessons of four dimensional instanton calculations teach us that exact results may be gleaned from constrained instantons, providing the approximate solutions are suitably compatible with supersymmetry (see for example [11]). We therefore identify approximate Bogomoln'yi type solutions to the equations of motion. These solutions are nothing more than the BPS vortices of the abelian Higgs model. In the remainder of section 3 we compute the various components necessary to perform the instanton computation and piece everything together to find that the isometry of the smeared NS5-brane is indeed broken, with the instanton corrections to the smeared function $H(r)$ taking the form,

$$H_{\rm inst}(r,\theta)=\frac{1}{g^2}+\frac{1}{2r}\left(1+\sum_{k=1}^{\infty}\sum_{\pm}m_k e^{-kr\pm ik\theta}\right) \tag{1.6}$$

where the numerical coefficients m_k encode certain information about the classical BPS vortex solutions. Due to the lack of integrability of the vortex equations, the results necessary to determine m_k from first principles are not available in the soliton literature. For example, neither the one-vortex solution, nor the two-vortex moduli space is known analytically, facts which impede calculation of m_1 and m_2 respectively. However, we may use the conjectured equality,

$$H_{\rm inst}(r,\theta)=H_{\rm sugra}(r,\theta)$$

to yield predictions for these quantities; namely $m_k=1$ for all k. For the $k=1$ instanton sector, we will show that $m_1=l_1^4/8$, where the coefficient l_1 characterizes the exponential radial decay of the Higgs field in the single vortex solution (see equation (3.13)). This number is known only through numerical studies [12, 13]. However, agreement with supergravity predicts $l_1=8^{1/4}$. It is heartening that this value agrees with [13] to within one part in a thousand (and is within 2% of the older numerical value quoted in [12]).

[1] To compare with the metric for the NS5-brane in flat space, define $\hat{\mathbf{r}}=\mathbf{r}/t$ and $\hat{\theta}=\theta/t$, and send $t\rightarrow 0$, keeping $\hat{\mathbf{r}}$ and $\hat{\theta}$ fixed.

For the higher instanton sectors, the numbers m_k do not have as simple an interpretation. They are given by

$$m_k = (2k)^{-3}\,\nu(\tilde{\mathcal{M}}_k) \tag{1.7}$$

where $\tilde{\mathcal{M}}_k$ is the centered moduli space describing the relative positions of k BPS vortices in the abelian Higgs model, and $\nu(\tilde{\mathcal{M}}_k)$ is the weighted integral of the Euler form over $\mathcal{M}_k$ (see equation (3.16)). The appearance of the Euler form for different soliton moduli spaces is not uncommon in instanton calculations. In particular, it was first encountered by Dorey, Khoze and Mattis in the context of three-dimensional gauge theories [14], and the calculation presented here closely follows this paper. One difference in the present case is that this integral is weighted by the functions l_k^4, which characterize the exponential radial decay of the k-vortex solution. It would be interesting to understand these functions in more detail.

Other approaches relating the NS5-brane to ALF spaces include an analysis of the 5-brane worldvolume dynamics [15], mirror symmetry in three-dimensional gauge theories [16], Nahm transforms [17] and D-brane probes [18].

2. Gauge theory, ALF spaces and T-duality

In this section we introduce the gauge theory of interest. We start by describing the full supersymmetric gauge theory in all its glory. We show that in the classical, low-energy limit, this gauge theory reduces to the sigma-model with torsion describing the NS5-brane smeared on a transverse circle, as discussed in the introduction. In section 2.2 we perform a T-duality transformation, and show that this result is equivalent to the Taub-NUT metric. Appendix A contains a discussion of the generalization to A_{N-1} ALF spaces.

2.1 The gauge theory

Let us start with a description of the gauge theory. It has $\mathcal{N} = (4,4)$ supersymmetry (or 8 non-chiral supercharges) in $d = 1+1$ dimensions, and may be thought of as the $\mathcal{N} = (4,4)$ extension of the theories discussed recently by Hori and Kapustin [7]. There are three superfield representations of the $\mathcal{N} = (4,4)$ supersymmetry algebra: a vector multiplet, a hypermultiplet and a twisted hypermultiplet. To construct our gauge theory, we need one of each.

The $\mathcal{N} = (4,4)$ vector multiplet contains a U(1) gauge field, two complex scalars ϕ and σ, and two Dirac fermions λ and $\tilde{\lambda}$. It will prove convenient to decompose our $\mathcal{N} = (4,4)$ superfields into $\mathcal{N} = (2,2)$ representations, resulting in an $\mathcal{N} = (2,2)$ U(1) vector multiplet V and a neutral chiral multiplet Φ. For the purposes of writing the gauge kinetic terms, one usually exchanges the vector multiplet in favor of a twisted chiral multiplet $\Sigma = \bar{D}_+ D_- V$. (For a detailed introduction to $\mathcal{N} = (2,2)$ theories and superspace notation, see [4, 6]). The non-derivative terms in the component expansion of these superfields are,

$$\text{Vector multiplet} \begin{cases} \Sigma = \sigma - i\sqrt{2}\theta^+\bar{\lambda}_+ - i\sqrt{2}\bar{\theta}^-\bar{\lambda}_- + \sqrt{2}\theta^+\bar{\theta}^-(D^3 - iF_{12}) + \cdots \\ \Phi = \phi + \sqrt{2}\theta^+\tilde{\lambda}_+ + \sqrt{2}\theta^-\tilde{\lambda}_- + \frac{1}{\sqrt{2}}\theta^+\theta^-(D^1 + iD^2) + \cdots \end{cases} \tag{2.1}$$

The hypermultiplet contains two complex scalars, q and $\tilde{q}$, which are again paired with two Dirac fermions, ψ and $\tilde{\psi}$. It decomposes into two chiral multiplets Q and $\tilde{Q}$ with charge $+1$ and -1 respectively under the U(1) gauge group. Again, the lowest components in the expansion read,

$$\text{Hypermultiplet}\begin{cases} Q = q + \sqrt{2}\theta^+\psi_+ + \sqrt{2}\theta^-\psi_- + \theta^+\theta^- F + \cdots \\ \tilde{Q} = \tilde{q} + \sqrt{2}\theta^+\tilde{\psi}_+ + \sqrt{2}\theta^-\tilde{\psi}_- + \theta^+\theta^-\tilde{F} + \cdots \end{cases}$$

Finally, the twisted hypermultiplet also contains four scalars and two Dirac fermions. This time the scalars naturally pair off into a triplet $\mathbf{r} = (r^1, r^2, r^3)$ and a singlet θ. We denote the fermions as χ and $\tilde{\chi}$. Under decomposition into $\mathcal{N} = (2,2)$ superfields, we have a chiral multiplet Φ and a twisted chiral multiplet Θ, each of which is uncharged under the gauge group. Once more, the component fields are given by,

$$\text{Twisted Hypermultiplet}\begin{cases} \Psi = (r^1 + ir^2) + \sqrt{2}\theta^+\chi_+ + \sqrt{2}\theta^-\chi_- + \theta^+\theta^- G + \cdots \\ \Theta = (r^3 + i\theta) - i\sqrt{2}\theta^+\bar{\tilde{\chi}}_+ - i\sqrt{2}\bar{\theta}^-\tilde{\chi}_- + \theta^+\bar{\theta}^-\tilde{G} + \cdots \end{cases}$$

With these conventions, the action takes a simple form in superspace notation. The kinetic terms for all fields arise from D-terms,

$$\mathcal{L}_D = \int d^4\theta\ \frac{1}{e^2}\left(\Sigma^\dagger\Sigma + \Phi^\dagger\Phi\right) + \frac{1}{g^2}\left(\ \Theta^\dagger\Theta + \Psi^\dagger\Psi\right) + Q^\dagger e^{2V}Q + \tilde{Q}^\dagger e^{-2V}\tilde{Q}$$

while the F-terms and twisted F-terms are given by,

$$\mathcal{L}_F = \int d\theta^+ d\theta^-\ \left(\tilde{Q}\Phi Q - \Psi\Phi\right), \qquad \mathcal{L}_{\tilde{F}} = -\int d\theta^+ d\bar{\theta}^-\ \Sigma\Theta$$

The cubic superpotential is familiar from all theories with 8 supercharges. The other two terms give the coupling between the vector multiplet and twisted hypermultiplet. These ensure that the fields $\mathbf{r}$ play the role of a triplet of dynamical Fayet-Iliopoulos (FI) parameters, while θ is a dynamical theta-angle.

In component form, the action splits neatly into kinetic, scalar potential and fermion mass and Yukawa terms,

$$S = \frac{1}{2\pi}\int d^2x\ \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{pot}} + \mathcal{L}_{\text{yuk}} \tag{2.2}$$

where the pre-factor of $1/2\pi$ is the usual normalization of the Polyakov action, and ensures that T-duality acts as (radius) $\to$ (radius)$^{-1}$, with no further numerical factors. We have,

$$\begin{aligned} \mathcal{L}_{\text{kin}} = &\frac{1}{e^2}\left(\frac{1}{2}F_{01}^2 - \frac{1}{2}|\partial\phi|^2 + \frac{1}{2}|\partial\sigma|^2 + i(\bar{\lambda}_+\partial_-\lambda_+ + \bar{\tilde{\lambda}}_+\partial_-\tilde{\lambda}_+ + \bar{\lambda}_-\partial_+\lambda_- + \bar{\tilde{\lambda}}_-\partial_+\tilde{\lambda}_-)\right) + \\ &+ \frac{1}{g^2}\left(-\frac{1}{2}|\partial\mathbf{r}|^2 - \frac{1}{2}(\partial\theta)^2 + i(\bar{\chi}_+\partial_-\chi_+ + \bar{\tilde{\chi}}_+\partial_-\tilde{\chi}_+ + \bar{\chi}_-\partial_+\chi_- + \bar{\tilde{\chi}}_-\partial_+\tilde{\chi}_-)\right) + \\ &+ \left(-|\mathcal{D}q|^2 - |\mathcal{D}\tilde{q}|^2 + i(\bar{\psi}_+\mathcal{D}_-\psi_+ + \bar{\tilde{\psi}}_+\mathcal{D}_-\tilde{\psi}_+ + \bar{\psi}_-\mathcal{D}_+\psi_- + \bar{\tilde{\psi}}_-\mathcal{D}_+\tilde{\psi}_-)\right) \end{aligned}$$

where our conventions are $\partial_\pm = \partial_0 \pm \partial_1$, and $\mathcal{D}q = \partial q - iAq$, and $\mathcal{D}\tilde{q} = \partial\tilde{q} + iA\tilde{q}$. The scalar potential is given by,

$$\begin{aligned} \mathcal{L}_{\text{pot}} = &-\frac{e^2}{2}\left(|q|^2 - |\tilde{q}|^2 - r^3\right)^2 - \frac{e^2}{2}\left|2\tilde{q}q - (r^1 + ir^2)\right|^2 + \theta F_{01} - \\ &- \left(|\phi|^2 + |\sigma|^2\right)\left(|q|^2 + |\tilde{q}|^2 + g^2\right) \end{aligned}$$

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The terms on the first line make manifest the roles played by the fields $\mathbf{r}$ and θ as FI parameters and theta angles respectively. In particular, it is clear from the topological nature of the θF coupling that the physics is invariant under shifts of $\theta \to \theta + 2\pi$. It is worth noting that the supersymmetric completion of these couplings includes the mass g of the vector multiplet scalar fields, which may be seen in the second line. Finally, the fermion masses and Yukawa couplings are given by,

$$
\begin{aligned}
\mathcal{L}_{\rm yuk} = &-\left(\chi_-\tilde{\lambda}_+ + \tilde{\lambda}_-\chi_+ + \bar{\tilde{\chi}}_+\lambda_- + \bar{\lambda}_+\tilde{\chi}_-\right) - \left(\bar{\tilde{\lambda}}_+\bar{\chi}_- + \bar{\chi}_+\bar{\tilde{\lambda}}_- + \bar{\lambda}_-\tilde{\chi}_+ + \bar{\tilde{\chi}}_-\lambda_+\right) + \\
&+ i\sqrt{2}q\left(\bar{\lambda}_-\bar{\psi}_+ - \bar{\lambda}_+\bar{\psi}_- + i\tilde{\lambda}_-\tilde{\psi}_+ + i\tilde{\psi}_-\tilde{\lambda}_+\right) - \sqrt{2}\sigma\left(\bar{\psi}_-\psi_+ - \bar{\tilde{\psi}}_-\tilde{\psi}_+\right) - \\
&- i\sqrt{2}q^\dagger\left(\psi_+\lambda_- - \psi_-\lambda_+ - i\bar{\tilde{\psi}}_+\bar{\tilde{\lambda}}_- - i\bar{\tilde{\lambda}}_+\bar{\tilde{\psi}}_-\right) - \sqrt{2}\sigma^\dagger\left(\bar{\psi}_+\psi_- - \bar{\tilde{\psi}}_+\tilde{\psi}_-\right) + \\
&+ i\sqrt{2}\tilde{q}\left(\bar{\lambda}_-\bar{\tilde{\psi}}_+ - \bar{\lambda}_+\bar{\tilde{\psi}}_- - i\tilde{\lambda}_-\psi_+ + i\psi_-\tilde{\lambda}_+\right) - \sqrt{2}\phi\left(\tilde{\psi}_-\psi_+ - \tilde{\psi}_+\psi_-\right) - \\
&- i\sqrt{2}\tilde{q}^\dagger\left(\tilde{\psi}_+\lambda_- - \tilde{\psi}_-\lambda_+ + i\bar{\psi}_+\bar{\tilde{\lambda}}_- - i\bar{\tilde{\lambda}}_+\bar{\psi}_-\right) - \sqrt{2}\phi^\dagger\left(\bar{\psi}_+\bar{\tilde{\psi}}_- - \bar{\psi}_-\bar{\tilde{\psi}}_+\right)
\end{aligned}
$$

The theory enjoys an $\mathrm{SU}(2)_R\times\mathrm{SO}(4)_R$ R-symmetry group, under which the vector multiplet scalars transform in the $(\mathbf{1},\mathbf{4})$, the hypermultiplet scalars transform in the $(\mathbf{2},\mathbf{1})$, and the twisted hypermultiplet scalars transform as $(\mathbf{1}+\mathbf{3},\mathbf{1})$. We have chosen to normalize fields such that the engineering dimension of the gauge coupling constant is $[e^2]=2$, which implies the vector multiplet scalars have $[\phi]=[\sigma]=1$, while all other scalar fields are dimensionless, $[q]=[\tilde{q}]=[\mathbf{r}]=[\theta]=0$. Importantly, the coefficient of the twisted hypermultiplet kinetic terms is also dimensionless: $[g^2]=0$.

The low-energy theory. The vacuum moduli space of the theory, determined by the zero locus of the scalar potential, is given by

$$
F_{01} = \sigma = \phi = 0\,, \qquad |q|^2 - |\tilde{q}|^2 = r^3\,, \qquad 2\tilde{q}q = r^1 + ir^2 \tag{2.3}
$$

Since e^2 is the only dimensionful parameter in the theory, low energies corresponds to $e^2\to\infty$, and the physics is described by a sigma-model on the vacuum moduli space (2.3). The metric on the moduli space is inherited from the kinetic terms which, in this case, receive contributions from both the hypermultiplet and twisted hypermultiplet. To see this, it is useful to introduce the gauge-variant field $\alpha = -2\arg(iq)$. After imposing the constraint (2.3), the hypermultiplet bosonic kinetic terms may then be written as,

$$
|\mathcal{D}q|^2 + |\mathcal{D}\tilde{q}|^2 = \frac{1}{4r}\partial\mathbf{r}\cdot\partial\mathbf{r} + \frac{r}{4}(\partial\alpha + 2A + \boldsymbol{\omega}\cdot\partial\mathbf{r})^2 \tag{2.4}
$$

where $\nabla\times\boldsymbol{\omega} = \nabla(1/r)$, and we have used the fact that, in vacuo, $r=|\mathbf{r}|=|q|^2+|\tilde{q}|^2$. We now chose to work in $\alpha=0$ gauge, and integrate out the gauge field A. In the strict $e^2\to\infty$ limit, the kinetic terms for A vanish, but the θdA term does not. We have,

$$
A_\mu = -\frac{1}{2}\boldsymbol{\omega}\cdot\partial_\mu\mathbf{r} + \frac{1}{2r}\epsilon_{\mu\nu}\partial^\nu\theta
$$

After substitution into the kinetic terms (2.4), and including the twisted hypermultiplet kinetic terms, we find that the bosonic sector of the low-energy effective action is given by a sigma-model with torsion,

$$\mathcal{L}_{\text{bosonic}} = \frac{1}{2}H(r)\left(\partial_\mu \mathbf{r}\partial^\mu \mathbf{r} + \partial_\mu\theta\partial^\mu\theta\right) + \frac{1}{2}\epsilon_{\mu\nu}\boldsymbol{\omega}\cdot\partial^\mu \mathbf{r}\,\partial^\nu\theta \tag{2.5}$$

where the function $H(r)$ is given by,

$$H(r) = \frac{1}{g^2} + \frac{1}{2r}$$

As discussed in the introduction, this is the target space metric and torsion describing an NS5-brane localized in the $\mathbf{R}^3$ parameterized by $\mathbf{r}$, but smeared along a transverse circle parameterized by θ. In the following section, we shall see how quantum effects alter this metric. However, let us first continue our examination of the classical low-energy effective action, turning now to the fermions. Working once more in the infra-red limit $e^2 \to \infty$, we notice that the vector multiplet fermions, λ and $\tilde{\lambda}$, become Grassmannian Lagrange multipliers, imposing the fermionic constraints,

$$\begin{aligned} \psi_+ &= \frac{1}{\sqrt{2r}}\left(iq\bar{\tilde{\chi}}_+ + \tilde{q}^\dagger\chi_+\right), & \tilde{\psi}_+ &= \frac{1}{\sqrt{2r}}\left(-q^\dagger\chi_+ + i\tilde{q}\bar{\tilde{\chi}}_+\right) \\ \psi_- &= \frac{1}{\sqrt{2r}}\left(iq\bar{\tilde{\chi}}_- - \tilde{q}^\dagger\chi_-\right), & \tilde{\psi}_- &= \frac{1}{\sqrt{2r}}\left(-q^\dagger\chi_- - i\tilde{q}\bar{\tilde{\chi}}_-\right) \end{aligned} \tag{2.6}$$

Substituting these into the fermionic kinetic terms, and summing the contribution from both the hypermultiplet and twisted hypermultiplet, we have

$$\mathcal{L}_{\text{fermionic}} = iH(r)\left(\bar{\chi}_+\partial_-\chi_+ + \bar{\chi}_-\partial_+\chi_- + \bar{\tilde{\chi}}_+\partial_-\tilde{\chi}_+ + \bar{\tilde{\chi}}_-\partial_+\tilde{\chi}_-\right) + \mathcal{O}(\chi^2\partial q)$$

This combines with the bosonic action (2.5) to yield a the first two terms of an $\mathcal{N} = (4,4)$ supersymmetric sigma-model with torsion. The full supersymmetric action for this theory was given in [19] and a nice summary may be found in [20]. The full action takes the form,

$$S_{\text{susy}} = \frac{1}{2\pi}\int d^2x\;\frac{1}{2}G_{ij}\partial_\mu X^i\partial_\nu X^j + \frac{1}{2}B_{ij}\epsilon_{\mu\nu}\partial^\mu X^i\partial^\nu X^j + iG_{ij}\bar{\Omega}^i D\Omega^j + \frac{1}{4}R_{ijkl}\bar{\Omega}^i\Omega^j\bar{\Omega}^k\Omega^l \tag{2.7}$$

where the covariant derivative is given by $D\Omega^j = \partial\Omega^j + \Omega^i\Gamma^j_{ik}\partial X^k$. The connection Γ in this expression is constructed with respect to the torsion $T = dB$, and the Riemann tensor R is similarly defined. Comparing to the terms above, we have the metric G given by equation (1.3), and the fields normalized as $X^i = (r^1, r^2, r^3, \theta)$ and $\Omega^i = (\chi_+, \chi_-, \tilde{\chi}_+, \tilde{\chi}_-)$.

2.2 T-duality and Taub-NUT space

In this section, we would like to perform T-duality on the periodic direction θ, and show that the low-energy physics is determined by a sigma-model without torsion, with the target space metric given by Taub-NUT of equation (1.1). We work with the full gauge theory (2.2) rather than the low-energy fields. To perform the duality, we first isolate the terms involving θ, and then introduce the auxiliary lagrangian,

$$\mathcal{L}_{\text{dual}} = \frac{1}{2g^2}C_\mu C^\mu - \epsilon_{\mu\nu}C^\mu A^\nu + \epsilon_{\mu\nu}\partial^\mu C^\nu\kappa$$

where the topological nature of the final term ensures that physics is invariant under 2π shifts of the Lagrange multiplier κ. Integrating out κ, we have $C_\mu = \partial_\mu \theta$, which returns us to the original lagrangian. If, however, we choose to integrate out C, we find instead

$$C_\mu = g^2 \epsilon_{\mu\nu}(\partial^\nu \kappa + A^\nu)$$

and, in terms of the dual field κ, the lagrangian becomes

$$\mathcal{L}_{dual} = \frac{g^2}{2}(\partial\kappa + A)^2$$

Notice that the presence of the θF coupling causes the dual field κ to transform transitively under the gauge group action,

$$\mathrm{U}(1): \quad q \to e^{i\gamma} q\,, \qquad \tilde{q} \to e^{-i\gamma}\tilde{q}\,, \qquad \kappa \to \kappa + \gamma \tag{2.8}$$

From a geometrical perspective, it is this transitive action which results in a stabilization of the $\mathbf{S}^1$ fiber at infinity, leading to an ALF, as opposed to ALE, space [21]. (This is discussed further in Appendix A. From the gauge theoretical perspective this construction was first described in [22]). Let us see explicitly how this occurs. After restricting to the vacuum moduli space (2.3), and rewriting the hypermultiplet kinetic terms as (2.4), the low-energy bosonic action in terms of the dual variable becomes,

$$\mathcal{L}_{TN} = \frac{1}{2}H(r)\partial\mathbf{r}\cdot\partial\mathbf{r} + \frac{r}{4}(\partial\alpha + 2A + \boldsymbol{\omega}\cdot\partial\mathbf{r}) + \frac{g^2}{2}(\partial\kappa + A)^2$$

Recall that $\alpha = -2\arg(iq)$, and thus has period 4π. It remains to divide by the gauge action (2.8). There are a number of equivalent ways to implement this. Following [21], we chose to set $A = 0$, and to work in terms of the gauge invariant quantity $\psi = \alpha - 2\kappa$, which also has period 4π. We can then re-write,

$$\frac{r}{4}(\partial\alpha + \boldsymbol{\omega}\cdot\partial\mathbf{r})^2 + \frac{g^2}{2}(\partial\kappa)^2 = \left(r + \frac{1}{2}g^2\right)\left(\partial\kappa + \frac{1}{2}r\left(r + \frac{1}{2}g^2\right)^{-1}(\partial\psi + \boldsymbol{\omega}\cdot\partial\mathbf{r})\right)^2 + \\ + \frac{1}{8}H^{-1}(r)\,(\partial\psi + \boldsymbol{\omega}\cdot\partial\mathbf{r})^2$$

In this form, the second term is gauge invariant while the first, being a total square, is simply lost upon taking the U(1) quotient. Thus, our final low-energy bosonic effective action is

$$\mathcal{L}_{TN} = \frac{1}{2}H(r)\partial\mathbf{r}\cdot\partial\mathbf{r} + \frac{1}{8}H^{-1}(r)\,(\partial\psi + \boldsymbol{\omega}\cdot\partial\mathbf{r})^2$$

which is indeed the sigma-model with Taub-NUT metric (1.1) as advertised.

While the previous discussion of T-duality involved only the bosonic fields, the extension to the supersymmetric theory is simple. A superfield derivation using Rocek-Verlinde transformations [23] in the $\mathcal{N} = (4,4)$ context may be found in [24].

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3. Worldsheet instantons

In the previous section we examined the gauge theory at the classical level and found dual descriptions of the low-energy physics. In one set of variables, this was the sigma-model on Taub-NUT space while, in the other, we found the NS5-brane, smeared on a transverse circle. In this section we examine the quantum corrections to the latter description. Before proceeding, we should make the usual disclaimer regarding the Coleman-Mermin-Wagner theorem [25]. In two dimensional theories, a quantum moduli space of vacua does not exist, and the groundstate wavefunction spreads over all classical vacuum states. Here we work in the Born-Oppenheimer spirit, in which high momentum modes are integrated out to reveal a low-energy description in terms of a sigma-model on the quantum corrected moduli space. This approach is relevant for comparison to supergravity. Motivated by this comparison and, in particular, the expression (1.5) describing a localized NS5-brane, we search for instanton solutions of the gauge theory. In the following section, we identify the relevant semi-classical configurations. In section 3.2 we compute the instanton measure and various other accessories necessary to perform the calculation. Finally, in section 3.3 we calculate the k-instanton contribution to the four-fermi vertex and translate this into the corrections to the metric.

3.1 Aspects of instantons

In the gauge theory approach to sigma models, worldsheet instantons appear as vortices [4, 26]. This has several advantages, among them the fact that vortices exist — and contribute to correlation functions — even when there are no two-cycles in the target space [4, 5]. Indeed, this is the case here. Although the second homology class of both Taub-NUT and the smeared NS5-brane metric is trivial, the gauge theory does have instanton sectors labelled by,

$$-\frac{1}{2\pi}\int F_{12} = k \in \mathbf{Z}$$

where F_{12} denotes the Wick rotation of the field strength F_{01} into euclidean space. However, although the requisite topology of the gauge theory exists, there are no finite action solutions to the equations of motion with the desired boundary conditions. To see this, let us start by choosing a classical vacuum from within the moduli space of vacua. We necessarily have $\phi = \sigma = 0$ and, without loss of generality, we may employ the $\mathrm{SU}(2)_R$ symmetry to further set $\tilde{q} = r^1 + ir^2 = 0$, while

$$|q|^2 = r^3 = \zeta \tag{3.1}$$

The field θ is left unconstrained. To perform a semi-classical calculation, one must search for an instanton solution which asymptotes to this vacuum. However, the gauge field strength F_{12} is a source for the massless field whose vacuum expectation value parameterizes the moduli space. Since this field is massless, far from a suitably localized instanton it must solve the Laplace equations in two-dimensions and therefore runs logarithmically at large distances. For this reason solutions to the equations of motion will not asymptote to the vacuum (3.1) for finite ζ. This problem has also been discussed in related models [27].

Of course, this behavior is unsurprising: it is simply a reflection of the Coleman-Mermin-Wagner theorem mentioned previously. Nevertheless, in order to proceed with our Born-Oppenheimer approximation, we must circumvent this obstacle.

Although the physics is somewhat different, formally the problem is reminiscent of four-dimensional instantons in Yang-Mills-Higgs theories. Recall that the presence of a vacuum expectation value, v, for the Higgs field implies that the Yang-Mills instanton will shrink to zero size in order to minimize its action. In this case a procedure known as "constrained instantons" [10] is employed which, in practice, involves expanding around the configurations which are solutions at $v = 0$. In supersymmetric theories, these approximate solutions retain all their BPS properties, which allows them to deliver exact, quantitative information (see, for example, [11] for various applications and reviews). In the present two-dimensional situation, the logarithmic divergence implies that any putative solution wishes to expand. To halt this, we need to find the analog of the quantity v, which we can tune to zero in order to recover solutions. We will now show that this quantity is the asymptotic radius of Taub-NUT space, g^2. To see this, let us firstly truncate the theory to allow only the gauge field, q and r^3 to vary over space-time. All other scalar fields are restricted to their classical expectation values and it can be checked that they will not further destabilize our solution. The equation of motion for r^3 in this background is,

$$\partial^2 r^3 = g^2 e^2 \left(|q|^2 - r^3\right) \tag{3.2}$$

In the limit $g^2 = 0$, it is consistent to keep $r^3 = \zeta$, even as $|q|^2$ moves out of vacuum. With this truncation, the bosonic euclidean action is given by,

$$S = \frac{1}{2\pi}\int d^2x \ \frac{1}{2e^2}F_{12}^2 + |\mathcal{D}q|^2 + \frac{e^2}{2}\left(|q|^2 - \zeta\right)^2 + i\theta F_{12}$$

which is simply the abelian Higgs model at critical coupling, together with a θ term for the gauge field. Note the factor of i which appears in front of this latter term after Wick rotation to euclidean space. As is well known, this theory admits BPS vortex solutions [28]. The first order equations of motion may be found by a judicious completion of the square,

$$S = \frac{1}{2\pi}\int d^2x \ \frac{1}{2e^2}\left(F_{12} \mp e^2(|q|^2 - \zeta)\right)^2 + |\mathcal{D}_1 q \mp i\mathcal{D}_2 q|^2 + (\mp\zeta + i\theta)F_{12}$$

where the upper (lower) sign is taken for $k > 0$ ($k < 0$). Throughout the remainder of this paper, we work with $k > 0$. Introducing the complex basis $z = x^1 + ix^2$, our Bogomoln'yi equations take the form,

$$F_{12} = e^2(|q|^2 - \zeta)^2, \qquad \mathcal{D}_{\bar{z}} q = 0 \tag{3.3}$$

and solutions to these equations have the action

$$S_k = k\zeta + ki\theta \tag{3.4}$$

To summarize, the instanton calculation is performed in the limit $g^2 \to 0$, and finite e^2. In contrast, the sigma-model limit is finite g^2, and $e^2 \to \infty$. We shall show that the final answer is independent of e^2, justfying the latter choice. In Appendix B, we show that the supergravity metric (1.4) yields a finite contribution to the four-fermi vertex of interest in the limit $g^2 \to 0$, thus also justifying the former choice.

3.2 The instanton measure

To compute the contributions of instantons to the low-energy effective action, we must first isolate the zero modes. The bosonic zero modes are given by the solutions to the linearized Bogomoln'yi equations (3.3),

$$\begin{aligned}\epsilon_{\mu\nu}\partial_\mu \delta A_\nu &= e^2(q^\dagger \delta q + q\delta q^\dagger)\\ \mathcal{D}_\mu \delta q - i\delta A_\mu q &= i\epsilon_{\mu\nu}\mathcal{D}_\nu \delta q + \epsilon_{\mu\nu}\delta A_\nu q\end{aligned}$$

which are augmented by a suitable gauge fixing condition which we take to be,

$$\partial_\mu \delta A_\mu = ie^2(q\delta q^\dagger - q^\dagger \delta q)$$

Using complex spacetime coordinates and defining $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$, these can be elegantly combined into a bosonic Dirac equation,

$$\Delta \begin{pmatrix} \delta A_z \\ \delta q \end{pmatrix} = 0 \qquad \text{with} \qquad \Delta = \begin{pmatrix} \frac{2i}{e^2}\bar{\partial} & -q^\dagger \\ q & i\mathcal{D} \end{pmatrix} \tag{3.5}$$

These equations were analyzed by E. Weinberg [29], who showed, using index theory, that there exist $2k$ normalizable, linearly independent zero modes $(\delta_a A_\mu, \delta_a q)$, for $a = 1, \dots, 2k$. Of these, two are Goldstone modes, arising from broken translational invariance, and given by

$$\delta_\nu A_\mu = F_{\nu\mu}\,, \qquad \delta_\nu q = \mathcal{D}_\nu q \qquad \nu = 1, 2 \tag{3.6}$$

while the remaining $2(k-1)$ are not generated by any symmetry. It can be shown that these remaining zero modes correspond to the decomposition of the k-vortex soliton into k single vortices with arbitrary positions given by the zeroes of the Higgs field [30]. We can introduce $2k$ collective coordinates, X^a, $a = 1, \dots, 2k$, which parameterize the multi-vortex moduli space $\mathcal{M}_k$. This space is endowed with a complete Kähler metric, given by the overlap of the zero modes [31],

$$g_{ab} = \frac{1}{2\pi}\int d^2x \frac{1}{2e^2}\delta_a A_\mu\, \delta_b A_\mu + \frac{1}{2}\delta_a q \delta_b q^\dagger + \frac{1}{2}\delta_a q^\dagger\, \delta_b q \tag{3.7}$$

where the complex structure of g is inherited from the complex structure of the two-dimensional worldsheet. The moduli space $\mathcal{M}_k$ decomposes metrically as

$$\mathcal{M}_k = \mathbf{R}^2 \times \tilde{\mathcal{M}}_k$$

where $\mathbf{R}^2$ is parameterized by X^μ, the center of mass of the vortices. The restriction of the metric g to the $\mathbf{R}^2$ factor can be easily calculated by substituting the zero modes (3.6) into the metric (3.7). It yields $\zeta k \delta_{\mu\nu}$, as expected for a soliton of "mass" ζk ("mass" becomes "action" for instantons). The centered moduli space $\tilde{\mathcal{M}}_k$ is parameterized by the relative positions Y^p, $p = 1, \dots, 2(k-1)$ of k vortices. We denote the metric on this $2(k-1)$ dimensional space as $\tilde{g}_{pq}$. Its analytic form is unknown, even for $k = 2$.

When performing the instanton calculation we must integrate over all these collective coordinates. The measure is obtained by changing variables in the path integral, and is given by [32],

$$\int d\mu_B = \frac{\zeta k}{2\pi}\int d^2X \prod_{p=1}^{2(k-1)} dY^p\ \frac{\det^{1/2}(\tilde{g})}{(2\pi)^{k-1}} \tag{3.8}$$

Fermion zero modes. The fermionic zero modes are related to the bosonic zero modes via the unbroken supersymmetry. To see this more explicitly, we examine the Dirac equations for the vector and hypermultiplet fermions in vortex background, in the limit $g^2 \to 0$,

$$\Delta \begin{pmatrix} \frac{i\bar{\lambda}_+}{\sqrt{2}} \\ \psi_- \end{pmatrix} = \Delta \begin{pmatrix} \frac{\tilde{\lambda}_+}{\sqrt{2}} \\ \bar{\tilde{\psi}}_- \end{pmatrix} = 0$$

$$\Delta^\dagger \begin{pmatrix} \frac{i\bar{\lambda}_-}{\sqrt{2}} \\ \psi_+ \end{pmatrix} = \Delta^\dagger \begin{pmatrix} \frac{-\tilde{\lambda}_-}{\sqrt{2}} \\ \bar{\tilde{\psi}}_+ \end{pmatrix} = 0$$

where the Dirac operator Δ is the same as that encountered in the analysis of the bosonic zero modes (3.5). The fermionic zero modes are related to the bosonic ones by $\lambda \sim \sqrt{2}\delta A_z$ and $\psi \sim \delta q$. Note that, while Δ has $2k$ zero modes, $\Delta^\dagger$ has none. To see this consider the action on an arbitrary complex doublet Y and define the norm,

$$\begin{aligned} \left\| \Delta^\dagger Y \right\| &= \left| \frac{2i}{e^2}\partial y_1 + q^\dagger y_2 \right|^2 + \frac{2}{e^2}|i\bar{\mathcal{D}}y_2 - qy_1|^2 \\ &= \left| \frac{2}{e^2}\partial y_1 \right|^2 + \frac{2}{e^2}|\bar{\mathcal{D}}y_2|^2 + \frac{2}{e^2}|qy_1|^2 + |q^\dagger y_2|^2 - \frac{2i}{e^2}\left(y_1 y_2^\dagger \mathcal{D}q - y_1^\dagger y_2 \bar{\mathcal{D}}q^\dagger \right) \end{aligned}$$

The last two terms vanish when evaluated on the background of the vortex, while the middle two ensure that $\Delta^\dagger Y = 0$ if and only if $Y = 0$. Thus the zero modes are carried by the pairs $(i\bar{\lambda}_+, \psi_-)^T$, $(\tilde{\lambda}_+, \bar{\tilde{\psi}}_-)^T$, $(i\lambda_-, \bar{\psi}_+)^T$, and $(-\bar{\tilde{\lambda}}_-, \tilde{\psi}_+)^T$. For example, the Goldstone bosons of equation (3.6) are related to the fermionic zero modes generated by the four broken supersymmetries,

$$\begin{aligned} \bar{\lambda}_+ &= \frac{1}{\sqrt{2}}F_{12}\alpha_1\,, & \lambda_- &= \frac{1}{\sqrt{2}}F_{12}\alpha_2\,, & \tilde{\lambda}_+ &= \frac{1}{\sqrt{2}}F_{12}\tilde{\alpha}_1\,, & \bar{\tilde{\lambda}}_- &= \frac{1}{\sqrt{2}}F_{12}\tilde{\alpha}_2 \\ \psi_- &= \bar{\mathcal{D}}q\alpha_1\,, & \bar{\psi}_+ &= \bar{\mathcal{D}}q\alpha_2\,, & \bar{\tilde{\psi}}_- &= \bar{\mathcal{D}}q\tilde{\alpha}_1\,, & \tilde{\psi}_+ &= \bar{\mathcal{D}}q\tilde{\alpha}_2 \end{aligned} \tag{3.9}$$

The fermionic measure for these broken supersymmetries is determined by calculating the overlap as for the bosonic case,[2] yielding

$$\int d\bar{\mu}_F = \int d^2\alpha\, d^2\tilde{\alpha} \left(\frac{1}{2}\zeta k\right)^{-2} \tag{3.10}$$

There are $4(k-1)$ further fermionic zero modes, related by unbroken supersymmetry to the $2(k-1)$ relative vortex positions. Let us denote the corresponding Grassmannian collective coordinates as β^p and $\tilde{\beta}^p$, with $p = 1, \ldots, 2(k-1)$, where the β's arise from the (λ, ψ) pairs, and the $\tilde{\beta}$'s from the $(\tilde{\lambda}, \tilde{\psi})$ pairs. As with their bosonic partners, each of these must also be integrated over, with the measure given by [14]

$$\int d\tilde{\mu}_F = \int \prod_{p=1}^{2(k-1)} d\beta^p d\tilde{\beta}^p \frac{1}{\det(\tilde{g})} \tag{3.11}$$

[2]The constant $\frac{1}{2}\zeta k$ differs by a factor of $1/2$ from normalization of the bosonic Goldstone modes. This can be traced to the fact that, under supersymmetry transformation, $\delta\psi = \sqrt{2}i\bar{\mathcal{D}}q\xi$, and thus $\alpha = \sqrt{2}\xi$, where ξ are the infinitesimal supersymmetry parameters

The action, determinants and long-distance behavior. While the constant part of the instanton action is given by (3.4), the instanton action can, in principle, also depend on the collective coordinates. This occurs when the zero modes discussed in the last section, each of which solves the linearized equations of motion, cannot all be simultaneously integrated to solutions of the full equations of motion. While this does not occur for the bosonic collective coordinates [30], it is commonplace for fermionic collective coordinates and, indeed, is even required by supersymmetry considerations [14]. (For a toy model, and a discussion of these issues, see [33]). To see this, note that the vortices discussed here preserve $\mathcal{N} = (2,2)$ non-chiral[3] supersymmetry on their worldvolume. The low-energy dynamics of these objects is described by a "0+0-dimensional sigma-model". Naively, one may imagine that, since sigma models have only derivative couplings, such an action is trivial. However, the non-chiral supersymmetric extensions of sigma-models also include non-derivative, four-fermi couplings [34]. These survive in the instanton action and, in the present case, are given by [14]

$$S_{4-\text{fermi}} = \frac{1}{4}\tilde{R}_{pqrs}\beta^p\beta^q\tilde{\beta}^r\tilde{\beta}^s \tag{3.12}$$

where $\tilde{R}$ is the Riemann tensor on the relative vortex moduli space $\tilde{\mathcal{M}}_k$.

In any instanton calculation, one must also integrate over the non-zero modes. In supersymmetric theories, the non-zero eigenvalues of the bosonic and fermionic operators around the background of a BPS instanton are guaranteed to coincide. In the case of four dimensional instantons, 't Hooft showed long ago that this is sufficient to ensure the cancellation of the one-loop determinants. However, this cancellation need not necessarily occur for operators with continuous spectra for, while the spectrum of eigenvalues must coincide, the density need not. Indeed, for instantons in three-dimensional gauge theories, it can be shown that the integration over non-zero modes leads to a finite, calculable contribution [36]. In the present case however, such fears are groundless as the exponential fall-off of the vortex tail ensures that the spectrum is suitably well-behaved [29] and the non-zero modes cancel.

Finally, we turn to the long-distance behavior of fields in the background of the instanton. While no explicit analytic solutions to the Bogomoln'yi equations (3.3) have been found, at distances large compared to all other length scales, the asymptotic form of the solutions is known [12]. Using polar coordinates, $z = \rho\exp(i\vartheta)$, the solution for a k-vortex configuration with center of mass at the origin, $X = 0$, becomes

$$|q|^2 \to \zeta\left(1 - l_k(Y^p,\vartheta)\sqrt{\frac{2\pi L}{\rho}}\exp\left(-\frac{\rho}{L}\right)\right) \tag{3.13}$$

[3]Non-chiral supersymmetry in space-time dimensions less than two, refers to the dimensional reduction of a non-chiral theory in two dimensions. To see that the theory on the vortex worldvolume is indeed non-chiral in the two-dimensional sense, it suffices to notice that the gauge theory described in section 2 can be lifted in a supersymmetric fashion to 5+1 dimensions, where the vortices have a 3+1 dimensional worldvolume.

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where the characteristic length scale of the vortex is

$$L = \frac{1}{\sqrt{2e^2\zeta}}$$

The functions $l_k(Y^p, \vartheta)$ characterize the exponential return to vacuum. The fact that the vortex tail is exponential, as opposed to polynomial, implies that these coefficients are functions of both the relative positions of the vortices, as well as the angular position on the complex plane, as indicated. Certain properties of these functions were recently studied in [37]. For a single vortex, l_1 is simply a numerical coefficient. It is not known analytically, but has been calculated numerically [12, 13]. The newer result of Speight [13] is[4]

$$l_1 \approx 1.683 \pm 0.001 \tag{3.14}$$

3.3 The calculation

We will now collect together all the pieces in order to compute the k-instanton contribution to the four-fermi correlation function,

$$\begin{aligned} G_4^{(k)}(x_1, x_2, x_3, x_4) &= \langle \bar{\psi}_+(x_1)\psi_-(x_2)\tilde{\psi}_+(x_3)\bar{\tilde{\psi}}_-(x_4)\rangle\Big|_{k-\text{instanton}} \\ &= \int d\mu_B d\bar{\mu}_F d\tilde{\mu}_F \bar{\psi}_+(x_1)\psi_-(x_2)\tilde{\psi}_+(x_3)\bar{\tilde{\psi}}_-(x_4)\ e^{-S_k - S_{4-\text{fermi}}} \end{aligned}$$

where the various components of this expression can be found in equations (3.4), (3.8), (3.10), (3.11) and (3.12). The fermions are replaced in the path-integral by their zero-mode values in the k-instanton background. In order to determine instanton contributions to higher dimension operators, one would need to know the explicit form for all such zero modes. However, for the special case of the four-fermi correlation function G_4, we need only the expression for the zero modes arising from unbroken supersymmetry (3.9). This is the semi-classical reflection of the non-renormalization theorems which ensure the calculability of two-derivative terms in theories with eight supercharges. From equations (3.9) and (3.13), we have the large distance expansion

$$\begin{aligned} \psi(x_i) &\to l_k(Y^p, \vartheta - \vartheta_i)\sqrt{\frac{\pi\zeta}{2L\rho}}e^{-\rho/L}e^{-i(k-1)(\vartheta-\vartheta_i)}\alpha \\ &= l_k(Y^p, \vartheta - \vartheta_i)\sqrt{\zeta}e^{-i(k-1)(\vartheta-\vartheta_i)}S_F\alpha \end{aligned}$$

Here S_F is the leading order behavior of the diagonal component of the fermionic propagator for a Dirac fermion of mass $1/L$ in the vacuum,

$$S_F \to \sqrt{\frac{\pi}{2L\rho}}\exp\left(-\frac{\rho}{L}\right)$$

[4]To compare with the conventions of [13], one must multiply by 2π. The older numerical result of [12] gives $l_1 \approx 1.708$.

as befits a Green's function for the Dirac operator $\Delta/2\pi$, defined in (3.5). Note that all dependence on the gauge coupling constant e^2 appears through the vortex length scale L which, in turn, appears only in the propagator. This is to be expected since gauge coupling constants cannot appear in the metric of the Higgs branch [38]. Note, in contrast, that not all position dependence can be absorbed in the propogator; angular dependence remains. This results in a form-factor for $G_4^{(k)}$ with $k > 1$ which can be converted to higher derivative interactions. To leading order in the derivative expansion, we have

$$G_4^{(k)}(x_1, x_2, x_3, x_4) = 2\zeta k^{-1}\pi^{-1}\nu(\tilde{M}_k)\, e^{-\zeta k - ik\theta} \int d^2X \prod_{i=1}^{4} S_F(X - x_i) \tag{3.15}$$

where the integration over all relative bosonic and fermionic zero modes has been collected into the function ν, and expressed as an integral over the $d = 2(k-1)$ dimensional moduli space $\tilde{\mathcal{M}}_k$ as in [14],

$$\begin{aligned} \nu(\tilde{M}_k) &= \frac{1}{(2\pi)^{d/2}} \int \prod_{p=1}^{d} dY^p d\beta^p d\tilde{\beta}^p d\vartheta \, \frac{l_k^4(Y^p, \vartheta)}{\sqrt{\det(\tilde{g})}} \frac{e^{-4i(k-1)\vartheta}}{2\pi} \exp\left(-\frac{1}{4}\tilde{R}_{pqrs}\beta^p\beta^q\tilde{\beta}^r\tilde{\beta}^s\right) \\ &= \frac{1}{(8\pi)^{d/2}(d/2)!} \int \frac{\prod_{p=1}^{d} dY^p}{\sqrt{\det(\tilde{g})}} \, \epsilon^{p_1p_2\ldots p_d}\epsilon^{q_1q_2\ldots q_d} \; \tilde{R}_{p_1p_2q_1q_2} \ldots \tilde{R}_{p_{d-1}p_dq_{d-1}q_d} \times \\ &\qquad \times \frac{1}{2\pi} \int d\vartheta \, l_k^4(Y^p, \vartheta)\, e^{-4i(k-1)\vartheta} \end{aligned} \tag{3.16}$$

For $k = 1$, we have $\nu_1 = l_1^4$. For $k > 1$, the expression above is familiar as the integral of the Euler form over the k-vortex relative moduli space, $\tilde{\mathcal{M}}_k$, weighted by a specific Fourier mode of $l_k^4(Y^p, \vartheta)$, the function which characterizes the exponential fall-off of the vortex solution.

Summing over all instanton sectors, $k \in \mathbf{Z}$, the correlation function $\sum_k G_4^{(k)}$ is equivalent to a four-fermi vertex for ψ in the low-energy effective action. Imposing the constraints (2.6) on this low-energy coupling, and re-expressing $\zeta \equiv r$, allows us to re-write the interaction in terms of the massless fermions χ,

$$\sum_{k=1}^{\infty} \xi_k \left(e^{-ik\theta}\bar{\tilde{\chi}}_+\tilde{\chi}_-\bar{\chi}_+\chi_- + e^{+ik\theta}\tilde{\chi}_+\bar{\tilde{\chi}}_-\chi_+\bar{\chi}_-\right) \tag{3.17}$$

where the coefficient is given by

$$\xi_k = \frac{1}{2^5\pi r k}\nu(\tilde{\mathcal{M}}_k)\; e^{-kr}$$

To compare to the supergravity prediction (1.5), we examine the low-energy action (2.7). The coefficient in front of the four-fermi term above is related to the Riemann tensor of the target space metric, computed with respect to the connection with torsion. In Appendix B we compute this object, at leading order in $1/r$, and to leading order in g^2, as is warranted given the discussion of section 3.1. Taking into account the symmetries of the Riemann tensor, the prediction for the low-energy four-fermi term has the same functional form as (3.17), but with the coefficient ξ_k replaced by $\tilde{\xi}_k$, given by

$$\tilde{\xi}_k = \frac{k^2}{4\pi r} e^{-kr}$$

We thus see that agreement with supergravity requires $\nu(\tilde{M}_k) = 8k^3$. It would be interesting to recover this prediction solely from the study of vortices. For now we claim agreement only for the $k = 1$ instanton sector, where we have $\nu_1 = l_1^4$. The supergravity prediction requires

$$l_1 = 8^{1/4} \approx 1.682$$

Encouragingly, this is in accord with the value (3.14) obtained through numerical studies of the vortex equations [12, 13].

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A. Linear sigma-model for ALF spaces

While the gauged linear sigma-model model for asymptotically locally euclidean (ALE) spaces is well known, the deformation to asymptotically locally flat (ALF) spaces appears to be less familiar to string theorists. For this reason we include this appendix describing the gauge theory whose Higgs branch is endowed with the hyperkähler metric on the ALF space with A_{N-1} singularity, given by,

$$ds^2 = H(r) d\mathbf{r} \cdot d\mathbf{r} + \frac{1}{4} H(r)^{-1} \left(d\psi + \boldsymbol{\omega} \cdot d\mathbf{r}\right)^2 \tag{A.1}$$

where $\mathbf{r}$ is a three-vector, and ψ has period 4π. The connection $\vec{\omega}$ is determined by $\nabla \times \vec{\omega} = 2\nabla H$, where H is the harmonic function,

$$H(r) = \frac{1}{g^2} + \frac{N}{2r} \tag{A.2}$$

For $N = 1$, this is the Taub-NUT metric of equation (1.1). The metric is written in Gibbons-Hawking coordinates which describe the ALF space as a fibration of $\mathbf{S}^1$ — parameterized by ψ — over an $\mathbf{R}^3$ base. Asymptotically $\mathbf{S}^1$ has radius g, and provides a

Hopf fibration over $\mathbf{S}^2 = \partial\mathbf{R}^3$, with winding number N. Rotations around this $\mathbf{S}^1$ yield a tri-holomorphic isometry which we denote as $\mathrm{U}(1)_F$,

$$\mathrm{U}(1)_F: \qquad \psi \to \psi + \alpha \tag{A.3}$$

It is instructive to consider the limit $g^2 \to \infty$, in which the boundary becomes the Lens space $\mathbf{S}^3/\mathbf{Z}_N$, and (A.2) is simply the flat metric on the ALE orbifold $\mathbf{C}^2/\mathbf{Z}_N$. We will refer to this as the ALE limit. In this case, there is a well-known gauged linear sigma model which reproduces this target space. It appears naturally in string theory as the theory on a probe D-brane [41]. It is the quiver theory with 8 supercharges — corresponding to $\mathcal{N} = (4,4)$ supersymmetry in two dimensions — associated to the affine A_{N-1} Dynkin diagram. The theory has gauge group $\mathcal{G} = \prod_{i=1}^N \mathrm{U}(k_i)$ where, for a single D-brane, $k_i = 1$ for all i. The matter content consists of a bi-fundamental hypermultiplet transforming under each pair of adjacent gauge groups, $(+k_i, -k_{i+1})$, where the index i is defined modulo N. Since the overall "center-of-mass" $\mathrm{U}(1)$ decouples, the interacting matter is,

$$A_{N-1} \text{ ALE Theory}: \qquad \mathrm{U}(1)^{N-1} \text{with } N \text{ hypermultiplets}$$

The Higgs branch of this theory reproduces the $\mathbf{C}^2/\mathbf{Z}_N$ orbifold. The bridge between the gauge theory and the geometry is provided by the hyperkahler quotient construction acting on the quaternionic space $\mathbf{H}^N$, parameterised by the N complex doublets w, which are the scalar components of the hypermultiplets. In the notation of section 2, we have $\omega_i^\dagger = (q_i^\dagger, \tilde{q}_i)$. The moment maps coincide with the D-terms of the gauge theory and are given by,

$$\boldsymbol{\mu}_i = \omega_i^\dagger \boldsymbol{\tau} \omega_i - \omega_{i+1}^\dagger \boldsymbol{\tau} \omega_{i+1} \tag{A.4}$$

where $\boldsymbol{\tau}$ are the Pauli matrices. Note that the sum of the moment maps is trivial, leaving only $(N-1)$ linearly independent triplets of constraints. After dividing by the $\mathrm{U}(1)^{N-1}$ gauge action, we arrive at the metric (A.2) in the ALE limit, $1/g^2 = 0$. It is important to note that the action of the $\mathrm{U}(1)_F$ isometry on the target space (A.3) arises from the flavor symmetry of the gauge theory,

$$\mathrm{U}(1)_F: \qquad \omega_i \to \exp(i\alpha)\, \omega_i \qquad \forall\ i \tag{A.5}$$

We would like to generalise this gauged linear sigma model to the ALF metric, with finite g^2. Geometrically, this requires squashing the $\mathbf{S}^1$ fibre at infinity. The hyperkähler quotient construction for such a space was discussed in [21], and a gauge theoretic interpretation (in the three-dimensional context) was given in [22]. The upshot of these papers is that the squashing from the ALE to ALF space can be achieved in a two-step process. Firstly, one gauges the $\mathrm{U}(1)_F$ isometry; secondly, this is coupled to a "linear multiplet". In the $d = 1+1$ dimensions of interest, a linear multiplet is also referred to as a twisted hypermultiplet.[5] Thus the matter content for the gauged linear sigma-model describing the ALF space is,

$$A_{N-1} \text{ ALF Theory}: \qquad \mathrm{U}(1)^N \text{with } N \text{ hypermultiplets and 1 twisted hypermultiplet}$$

[5] In $d = 2+1$ dimensions, the linear multiplet is also known as a twisted vector multiplet, and couples to $\mathrm{U}(1)_F$ through a Chern-Simons interaction. In $d = 3+1$, the linear multiplet contains a two-form field and three scalars, and couples to $\mathrm{U}(1)_F$ through a "BF" interaction.

For $N = 1$ this is the theory described in section 2. Let us denote the gauge field appearing in the additional vector multiplet as A_F. We further decompose the four scalars of the twisted hypermultiplet as a triplet $\mathbf{r}$ and a singlet θ. The full couplings between the vector and twisted hypermultiplet is given in section 2.1. Here we isolate the terms relevant for the hyperkähler quotient. They include the terms,

$$\Delta\mathcal{L} = \frac{1}{2e^2} dA_F^2 + \frac{1}{2g^2}\left(d\theta^2 + d\mathbf{r}^2\right) + \theta \wedge dA_F \tag{A.6}$$

We have introduced the coupling constants e^2 and g^2. The metric on the Higgs branch cannot depend upon coupling constants for vector multiplets [38] and, to truly restrict to the Higgs branch, we take the $e^2 \to \infty$ limit. However, the metric on the Higgs branch does depend on g^2. Indeed, this will determine the amount of asymptotic squashing of the $\mathbf{S}^1$ fiber through its contribution to the harmonic function (A.2). The key to seeing this fact lies in understanding the supersymmetric completion of $\theta \wedge dA_F$ term in (A.6). As well as various fermion couplings, there are also further D-terms which enhance the moment map for the $\mathrm{U}(1)_F$ gauge action to include a coupling to $\mathbf{r}$,

$$\boldsymbol{\mu}_F = \sum_{i=1}^{N} w_i^\dagger \boldsymbol{\tau} w_i - \mathbf{r} \tag{A.7}$$

The presence of the $\mathbf{r}$ term in the moment map implies the presence of a field transforming transitively under $\mathrm{U}(1)_F$ [21]. As explained in detail in section 2, such a field exists, but appears only after dualizing θ in exchange for a new scalar field κ through the relationship $d\theta = g^2\, {}^\star(d\kappa + A)$, after which the kinetic terms require,

$$\mathrm{U}(1)_F : \qquad \sigma \to \sigma + \alpha \tag{A.8}$$

The explicit hyperkähler quotient reduction with this moment map was performed in [21] where it was shown to reproduce the metric (A.1). The case of $N = 1$ was described in detail in section 2.2.

B. The Riemann tensor

In this appendix we calculate the leading order contribution from all sectors of instantons and anti-instantons to the Riemann tensor for the metric

$$ds^2 = H(r,\theta) = H(r)\left(d\mathbf{r}\cdot\mathbf{dr} + \mathbf{d\theta^2}\right)$$

where $\mathbf{r} = (r^1, r^2, r^3)$, and H depends only on θ and $|\mathbf{r}| = r$,

$$H(r,\theta) = \frac{1}{g^2} + \frac{1}{2r}\frac{\sinh r}{\cosh r - \cos\theta} \tag{B.1}$$

Setting $r^4 \equiv \theta$, the torsion is given by

$$T_{ijk} = \epsilon^l_{ijk}\partial_l H^{-1} \tag{B.2}$$

We define the vierbein one-forms $e^\alpha = e_i^\alpha dr^i$,

$$e^\alpha = H^{1/2} dr^i \delta_i^\alpha$$

which facilitates calculation of the spin connection, ω

$$\omega^\alpha_\beta = \frac{1}{2}\left(H^{-1}\partial_j H\, dr^i - T^i{}_{jk} dr^k\right)\delta_i^\alpha \delta_\beta^j - \frac{1}{2}H^{-1}\partial_i H\, dr^j \delta^{i\alpha}\delta_{j\beta}$$

It can be checked that these reproduce the Levi-Civita connection when $T \equiv 0$, and satisfy Cartan's first structure equation,

$$de^\alpha + \omega^\alpha_\beta \wedge e^\beta = T^\alpha \equiv \frac{1}{2}T^\alpha_{\beta\gamma} e^\beta \wedge e^\gamma$$

The curvature two-forms are now determined using Cartan's second equation,

$$d\omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \omega^\gamma_\beta = R^\alpha_\beta \equiv \frac{1}{2}R^\alpha{}_{\beta\gamma\delta} e^\gamma \wedge e^\delta$$

From which we may extract the Riemann tensor which, in the original r^i coordinates, reads

$$\begin{aligned} R^i_{jkl} = &\left(\frac{1}{4}H^{-2}\partial_j H \partial_{[l} H - \frac{1}{4}H^{-1}\partial_k H T^k_{j[l} - \frac{1}{2}\partial_j(H^{-1}\partial_{[l}H)\right)\delta^i_{k]} - \\ &- \left(\frac{1}{4}H^{-2}\partial_i H \partial_{[l} H + \frac{1}{4}H^{-1}\partial_k H T^i_{m[l}\delta^{km} - \frac{1}{2}\partial_i(H^{-1}\partial_{[l}H)\right)\delta_{k]j} - \\ &- \frac{1}{2}H^{-1}T^i_{lk}\partial_j H + \frac{1}{4}T^m_{k[j}\delta_{l]m}H^{-1}\partial_i H + \frac{1}{2}\partial_{[l}T^i_{k]j} - \frac{1}{4}T^i_{m[l}T^m_{k]j} - \\ &- \frac{1}{2}H^{-2}(\partial_m H)^2 \delta^i_{[k}\delta_{j]l} \end{aligned}$$

To compare with the instanton calculation, we should expand this expression for large r. In fact, from (B.1), we see that there exist two such expansion paramters: r^{-1} and e^{-r}. We wish to keep all orders in the instanton expansion, but only leading order in r^{-1}. Expanding the function (B.1), we have

$$\partial H \to \frac{1}{2r}\sum_{k=1}^{\infty}\sum_{\pm} k\, e^{-kr \pm ik\theta} \tag{B.3}$$

from which we deduce that the leading order contribution to the Riemann tensor appears at $1/r$, and is given by the eminently more managable

$$R^i_{jkl} \to -\frac{1}{2}H^{-1}\partial_j\partial_{[l}H\delta^i_{k]} + \frac{1}{2}H^{-1}\partial_i\partial_{[l}H\delta_{k]j} - \frac{1}{2}\partial_{[l}T^i_{k]j}$$

Finally, following the discussion of the constrained instantons presented in section 3, we also wish to expand the Riemann tensor in powers of $g^2 \ll 1$. Since the first two terms are of order g^2, while the third is of order g^6, we neglect the torsion completely:

$$R^i_{jkl} \to -\frac{1}{2}H^{-1}\partial_j\partial_{[l}H\delta^i_{k]} + \frac{1}{2}H^{-1}\partial_i\partial_{[l}H\delta_{k]j}$$

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To compare with the instanton computation, we change to a complex basis of coordinates,

$$z^1 = r^1 + ir^2\,, \qquad z^2 = r^3 + i\theta$$

and compute the components of the Riemann tensor evaluated at the point $z^1 = 0$. At leading order in r^{-1} and g^2, we find

$$\begin{aligned}
R^{\bar{1}}_{212}\Big|_{z^1=0} &= R^{\bar{1}}{}_{2\bar{1}\bar{2}}\Big|_{z^1=0} \to 0 \\
R^{1}{}_{212}\big|_{z^1=0} &\to -\frac{g^2}{2r}\sum_{k=1}^{\infty} k^2 e^{-kr-ik\theta} \\
R^{\bar{1}}{}_{\bar{2}\bar{1}\bar{2}}\Big|_{z^1=0} &\to -\frac{g^2}{2r}\sum_{k=1}^{\infty} k^2 e^{-kr+ik\theta}
\end{aligned} \tag{B.4}$$

To compare with the appearance of this Riemann tensor in the low-energy effective action (2.7), we must lower the sole upper index. At leading order in r, this simply removes the factor of g^2, and the relevant components of the tensor are,

$$\begin{aligned}
R_{\bar{1}212}\big|_{z^1=0} &\to -\frac{1}{2r}\sum_{k=1}^{\infty} k^2 e^{-kr-ik\theta} \\
R_{1\bar{2}\bar{1}\bar{2}}\big|_{z^1=0} &\to -\frac{1}{2r}\sum_{k=1}^{\infty} k^2 e^{-kr+ik\theta}
\end{aligned}$$

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